FAST APPROXIMATION OF THE LAMBERT W FUNCTION FOR VIRTUAL ANALOG MODELLING

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WHAT THIS PAPER IS ABOUT

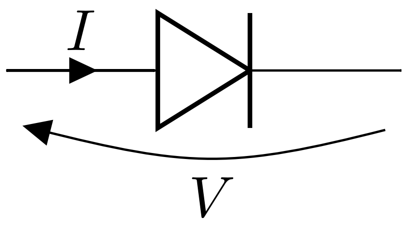
Solving simple transcendental equations

that arise in virtual analog modelling

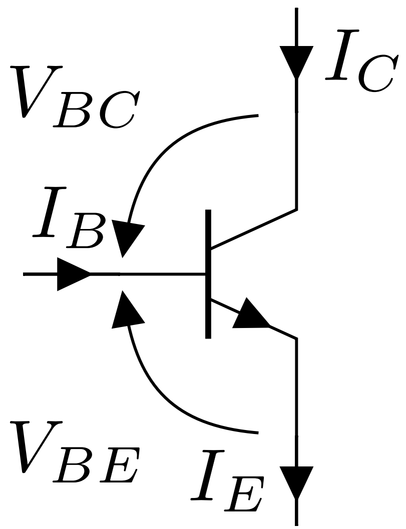
as fast as possible

and show two real world applications.

DIODES AND TRANSISTORS



$$I = I_{\mathrm{s}} \left(e^{\frac{V}{n V_{\mathrm{T}}}} - 1 \right)$$



$$I_{\mathrm{C}} = I_{\mathrm{s}} \left[\left(e^{\frac{V_{\mathrm{BE}}}{V_{\mathrm{T}}}} - e^{\frac{V_{\mathrm{BC}}}{V_{\mathrm{T}}}} \right) - \frac{1}{\beta_{\mathrm{r}}} \left(e^{\frac{V_{\mathrm{BC}}}{V_{\mathrm{T}}}} - 1 \right) \right]$$

$$I_{\mathrm{E}} = I_{\mathrm{s}} \left[\left(e^{\frac{V_{\mathrm{BE}}}{V_{\mathrm{T}}}} - e^{\frac{V_{\mathrm{BC}}}{V_{\mathrm{T}}}} \right) + \frac{1}{\beta_{\mathrm{f}}} \left(e^{\frac{V_{\mathrm{BE}}}{V_{\mathrm{T}}}} - 1 \right) \right]$$

$$I_{\mathrm{B}} = I_{\mathrm{E}} - I_{\mathrm{C}}$$

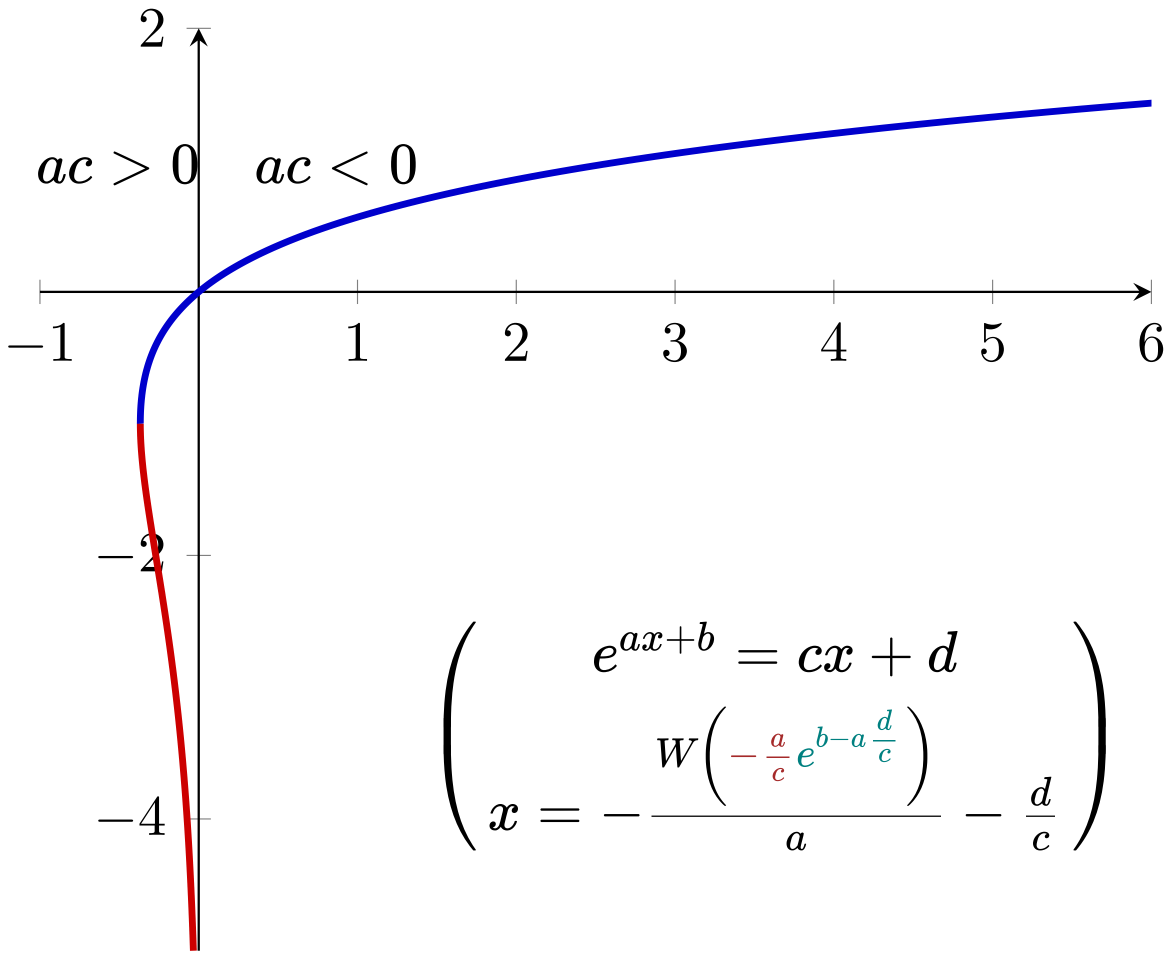
SOLVING SIMPLE TRANSCENDENTAL EQUATIONS

General form: $e^{ax+b} = cx + d$

Lambert W function: $x = f^{-1}\left(xe^{x}\right) = W\left(xe^{x}\right)$

Analytical solution: $x = -\frac{W\left(-\frac{a}{c}e^{b-a\frac{d}{c}}\right)}{a} - \frac{d}{c}$

LAMBERT W FUNCTION: $y = W(x)$



$$\left(\begin{array}{c} e^{ax+b} = cx + d \\ x = -\dfrac{W\left(-\frac{a}{c} e^{b - a\frac{d}{c}}\right)}{a} - \dfrac{d}{c} \end{array} \right)$$

PRACTICAL CONSIDERATIONS

$$e^{ax+b} = cx + d$$

$$x = -\frac{W\left(-\frac{a}{c}e^{b-\frac{a}{c}d}\right)}{a} - \frac{d}{c}$$

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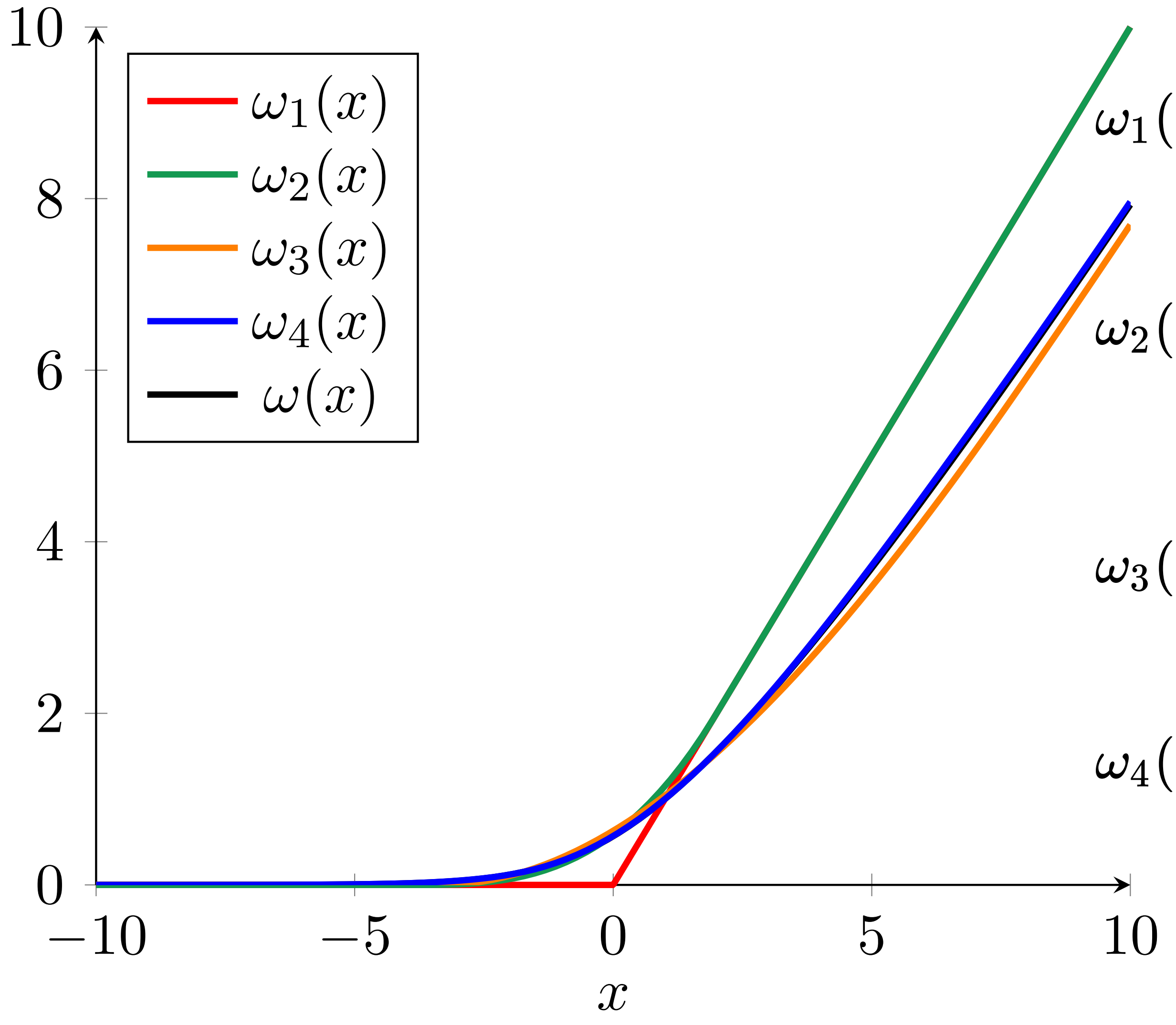
if $ac < 0$: $x = -\frac{W\left(e^{b-\frac{a}{c}d+\log\left(-\frac{a}{c}\right)}\right)}{a} - \frac{d}{c}$

Wright ω function: $\omega(x) = W\left(e^{x}\right)$

$$x = -\frac{\omega\left(b-\frac{a}{c}d+\log\left(-\frac{a}{c}\right)\right)}{a} - \frac{d}{c}$$

WRIGHT ω FUNCTION: $y = \omega(x)$



$$\omega_1(x) = \max(0, x)$$

$$\omega_2(x) = \begin{cases} 0 \\ \alpha x^3 + \beta x^2 + \gamma x + \zeta \\ x \end{cases}$$

$$\omega_3(x) = \begin{cases} 0 \\ \alpha x^3 + \beta x^2 + \gamma x + \zeta \\ x - \log(x) \end{cases}$$

$$\omega_4(x) = \omega_3(x) - \frac{\omega_3(x) - e^{x - \omega_3(x)}}{\omega_3(x) + 1}$$

FAST $\log(x)$ (EXPLOITING FP REPRESENTATION)

FP representation: $x = S2^E(1 + M)$

where $S = \mathrm{sgn}(x)$, $E \in \mathbb{Z}$, $M \in [0, 1)$.

Expressing $\log(x) = \frac{1}{\log_2(e)}(E + \log_2(1 + M))$

we can approximate $\log_2(x) \approx \alpha x^3 + \beta x^2 + \gamma x + \zeta$

since $1 + M \in [1, 2)$

Total: 3 bitwise op, 2 int sum, 1 int-to-FP, 4 FP mul, 4 FP sum

FAST e^x (EXPLOITING FP REPRESENTATION)

FP representation: $x = S2^E(1 + M)$

where $S = \mathrm{sgn}(x), E \in \mathbb{Z}, M \in [0, 1)$.

Express $e^x = 2^{\lfloor y \rfloor} 2^{y - \lfloor y \rfloor}, \quad y = \frac{1}{\log(2)} x$

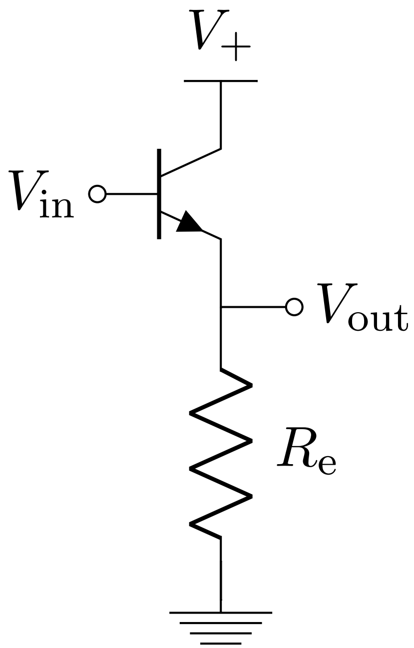
FP representation of $2^{\lfloor y \rfloor} \rightarrow S = 1, E = \lfloor y \rfloor, M = 0$

Since $y - \lfloor y \rfloor \in [0, 1)$

we can approximate $2^x \approx \alpha x^3 + \beta x^2 + \gamma x + \zeta$

Total: 2 if, 1 bitwise op, 2 int sum, 1 FP-to-int, 1 int-to-FP, 5 FP mul, 4 FP sum

COMMON COLLECTOR VOLTAGE BUFFER



$$I_{\text{s}}\left(e^{\frac{V_{\text{in}}-V_{\text{out}}}{V_{\text{T}}}} - e^{\frac{V_{\text{in}}-V_+}{V_{\text{T}}}} + \frac{e^{\frac{V_{\text{in}}-V_{\text{out}}}{V_{\text{T}}}}-1}{\beta_{\text{f}}}\right) = \frac{V_{\text{out}}}{R_{\text{e}}}$$

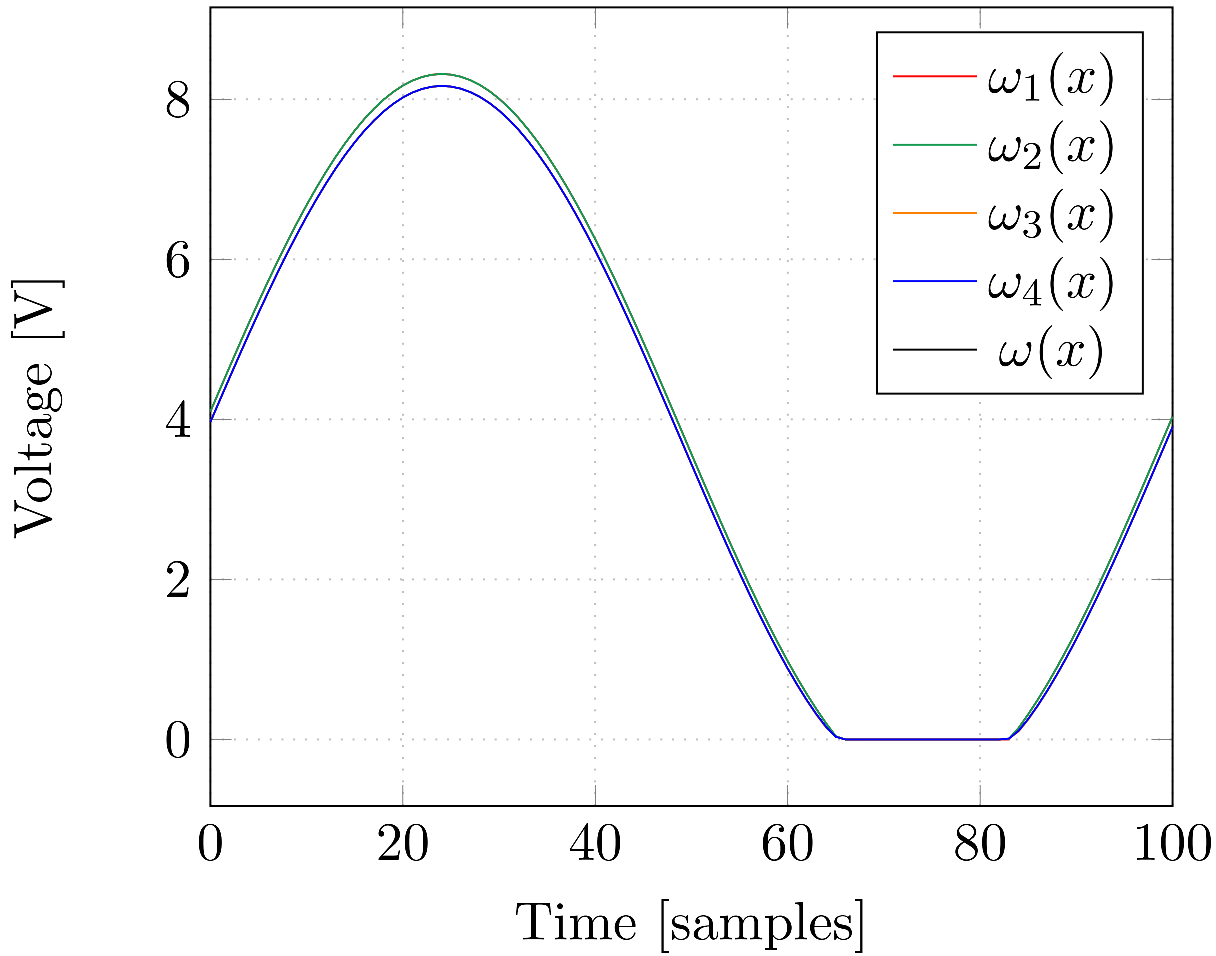
$$V_{\text{out}} = V_{\text{T}}\omega\left(\frac{V_{\text{in}}+V_{\text{x}}}{V_{\text{T}}} + k\right) - V_{\text{x}},$$

$$V_{\text{x}} = I_{\text{s}}R_{\text{e}}\left(e^{\frac{V_{\text{in}}-V_+}{V_{\text{T}}}} + \frac{1}{\beta_{\text{f}}}\right),$$

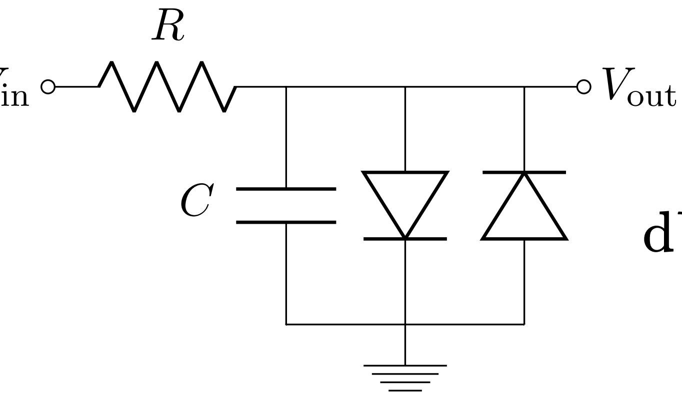
$$k = \log\left(\frac{I_{\text{s}}R_{\text{e}}}{V_{\text{T}}}\left(1+\frac{1}{\beta_{\text{f}}}\right)\right).$$

Total: 4 sum, 4 mul, 1 exp, 1 ω

COMMON COLLECTOR VOLTAGE BUFFER OUTPUT



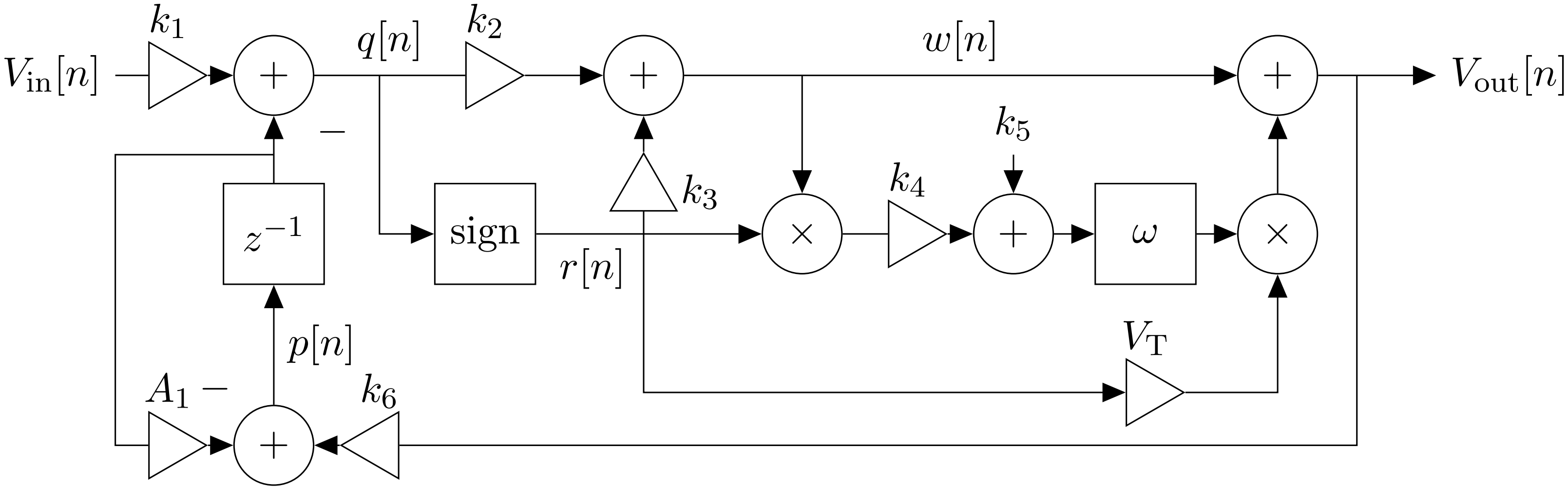
DIODE CLIPPER



$$\frac{\mathrm{d}V_{\mathrm{out}}}{\mathrm{d}t} = \frac{V_{\mathrm{in}} - V_{\mathrm{out}}}{RC} - 2\frac{I_{\mathrm{s}}}{C}\sinh\left(\frac{V_{\mathrm{out}}}{V_{\mathrm{T}}}\right)$$

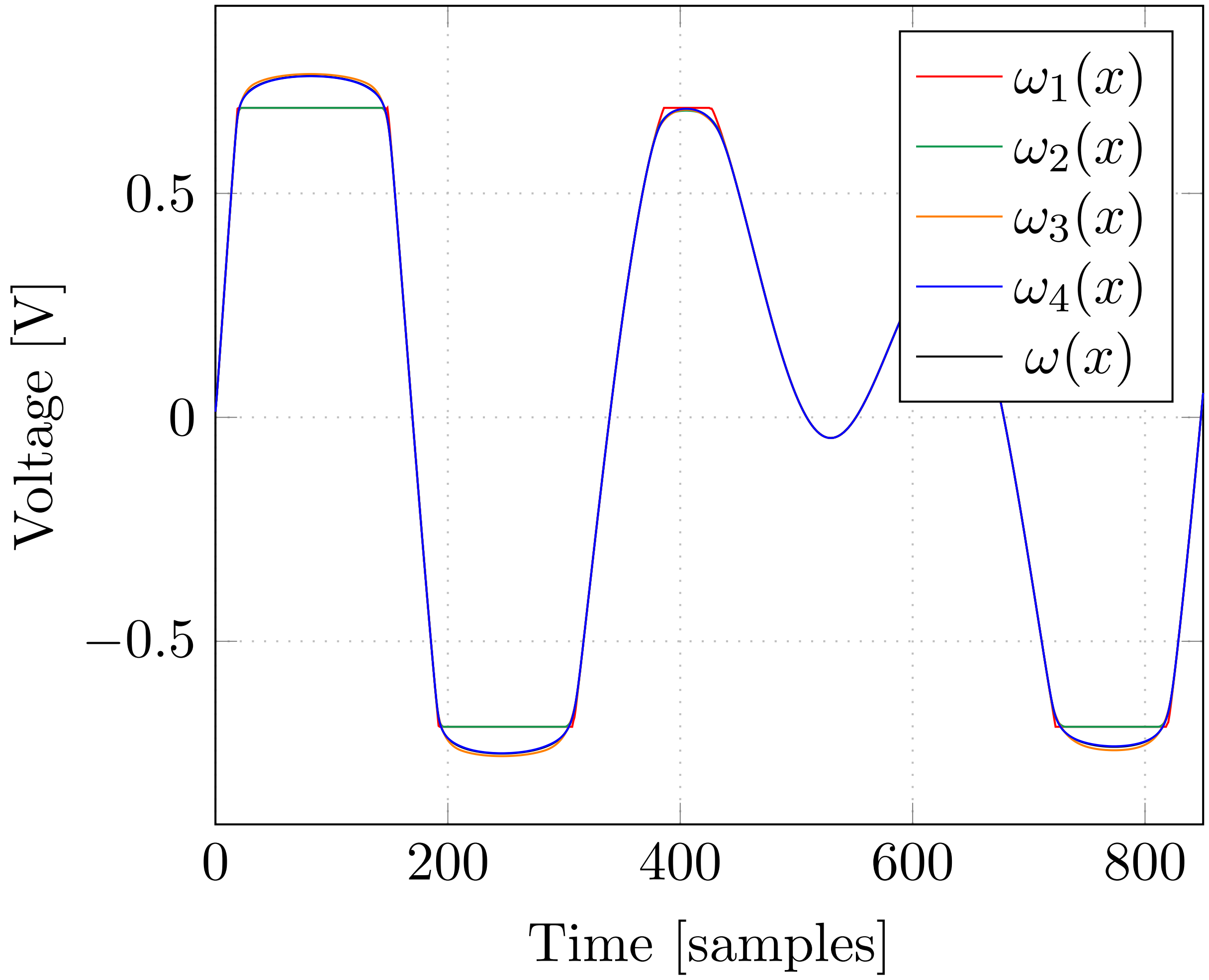
$$\mathrm{d}V_{\mathrm{out}}[n] = B_0 V_{\mathrm{out}}[n] + B_1 V_{\mathrm{out}}[n-1] - A_1 \mathrm{d}V_{\mathrm{out}}[n-1]$$

$$\sinh(x) \approx \frac{1}{2}\mathrm{sgn}(x)\left(e^{|x|} - 1\right)$$



Total: 5 sum, 9 mul, 1 sign, 1 ω

DIODE CLIPPER OUTPUT



CONCLUSIONS

1. Use $W\left(ke^{x}\right) = W\left(e^{x+\log(k)}\right) = \omega\left(x + \log(k)\right)$

2. We have fast $\omega(x)$ (and $\log(x)$, e^{x}) approximations

3. Grab code at http://dangelo.audio/dafx2019-omega.html